Exam Solutions

1. (a) The series is a convergent geometric series and

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{4^{k+1}}{5^{k-1}} &= \sum_{k=0}^{\infty} (4 \cdot 5) \left(\frac{4}{5}\right)^k = \sum_{k=0}^{\infty} 20 \left(\frac{4}{5}\right)^k \\ &= \frac{20}{1 - \frac{4}{5}} = 5 \cdot 20 = 100.\end{aligned}$$

   (b) Let $a_k = \dfrac{k}{7^k}$. Applying the ratio test we find that

$$\begin{aligned}\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| &= \lim_{k\to\infty} \left| \frac{\frac{k+1}{7^{k+1}}}{\frac{k}{7^k}} \right| = \lim_{k\to\infty} \left| \frac{7^k(k+1)}{7^{k+1}k} \right| \\ &= \lim_{k\to\infty} \frac{1}{7} \cdot \frac{k+1}{k} = \frac{1}{7}.\end{aligned}$$

   Thus the series is convergent for $|x| < 7$.

   Furthermore, as the ratio test does not tell us anything about the convergence at the endpoints where $|x| = 7$ we must check them separately:

   - When $x = 7$ we find that

$$\sum_{k=1}^{\infty} \frac{k}{7^k} x^k = \sum_{k=1}^{\infty} \frac{k}{7^k} 7^k = \sum_{k=1}^{\infty} k$$

     which is clearly divergent.
   - Likewise, when $x = -7$ we get

$$\sum_{k=1}^{\infty} \frac{k}{7^k} x^k = \sum_{k=1}^{\infty} \frac{k}{7^k} (-7)^k = \sum_{k=1}^{\infty} (-1)^k k$$

     which also obviously diverges.

   Thus we conclude that the series converges when $-7 < x < 7$ and diverges otherwise.

2. (a) The 3rd degree Maclaurin polynomial for the sine function is

$$P_3(x) = x - \frac{x^3}{3!} = x - \frac{x^3}{6}$$

and the 2nd degree Maclaurin polynomial for the cosine function is

$$P_2(x) = 1 + \frac{x^2}{2!} = 1 - \frac{x^2}{2}.$$

Thus this gives us the approximation

$$\cos(\sin(0.1)) = 1 - \frac{1}{2}\left(0.1 - \frac{(0.1)^3}{6}\right)^2 = 0.99501665\ldots$$

For comparison the exact value is $0.99502078\ldots$.

(b) First we note that

$$\lim_{x\to 0}(e^{ax} - 1 + x^2) = 1 - 1 + 0 = 0$$

and

$$\lim_{x\to 0}\ln(1 + 2ax) = \ln(1 + 0) = \ln 1 = 0.$$

Since a direct substitution would lead to an indeterminate form "0/0" we may apply L'Hospital's rule to determine the limit. The derivative of the numerator is

$$\frac{d}{dx}(e^{ax} - 1 + x^2) = ae^{ax} + 2x$$

and the derivative of the denominator is

$$\frac{d}{dx}\ln(1 + 2ax) = \frac{2a}{1 + 2x}.$$

So supposing that $a \neq 0$ we find that the limit is

$$\begin{aligned}\lim_{x\to 0}\frac{e^{ax} - 1 + x^2}{\ln(1 + 2ax)} &= \lim_{x\to 0}\frac{ae^{ax} + 2x}{\frac{2a}{1+2x}}\\ &= \lim_{x\to 0}\frac{(ae^{ax} + 2x)(1 + 2x)}{2a}\\ &= \frac{(ae^{0\cdot x} + 2\cdot 0)(1 + 2\cdot 0)}{2a} = \frac{a}{2a} = \frac{1}{2}\end{aligned}$$

and conclude that the limit does indeed not depend on the value of the parameter a.

3. (a) As $x^4 \geq 0$ for all $x \in \mathbf{R}$ it follows that

$$f'(x) = 3 + 20x^4 \geq 3 + 0 = 3 > 0$$

for all $x \in \mathbf{R}$ and hence the function f is strictly increasing.

(b) First we notice that

$$f(1) = 3 \cdot 1 + 4 \cdot 1^5 = 7 \;\Leftrightarrow\; f^{-1}(7) = 1$$

and so

$$(f^{-1})'(7) = \frac{1}{f'(f^{-1}(7))} = \frac{1}{f'(1)} = \frac{1}{3 + 20 \cdot 1^4} = \frac{1}{23}.$$

4. First, let $x = t^2$. It follows that $dx = 2t\,dt$ and for $x \geq 0$ we find that

$$t = \sqrt{x} \;\Rightarrow\; \begin{cases} x \to 0 & \Rightarrow t \to 0 \\ x \to \infty & \Rightarrow t \to \infty \end{cases}$$

Hence the substitution transforms the integral as

$$\int_0^\infty e^{-\sqrt{x}}\,dx = \int_0^\infty e^{-t}(2t)\,dt.$$

Next, let $f'(t) = e^{-t}$ and $g(t) = 2t$. Applying Integration by Parts we get

$$\begin{aligned} \int_0^\infty e^{-t}(2t)\,dt &= \int_0^\infty f'(t)g(t)\,dt \\ &= \lim_{a\to\infty} f(a)g(a) - f(0)g(0) - \int_0^\infty \underbrace{f(t)g'(t)}_{=-2e^{-t}}\,dt \\ &= \underbrace{\lim_{a\to\infty} (2a \cdot (-e^{-a}))}_{=0} - \underbrace{(2 \cdot 0 \cdot (-e^{-0}))}_{=0} + \int_0^\infty 2e^{-t}\,dt \\ &= \underbrace{\lim_{a\to\infty} (-2e^{-a})}_{=0} - \underbrace{(-2e^{-0})}_{=-2} = 2. \end{aligned}$$

Thus

$$\int_0^\infty e^{-\sqrt{x}}\,dx = 2.$$

5. (a) From the course materials we know (or e.g. solving by separation of variables) that

$$y' = ky \;\Rightarrow\; y(t) = Ce^{kt}.$$

Here the value of the constant C is determined by the initial condition $y(0) = 100$ as

$$y(0) = Ce^{k\cdot 0} = C = 100.$$

It then follows from the second condition $y(2) = 10^8$ that

$$y(2) = 100e^{2k} = 10^8 \;\Rightarrow\; e^{2k} = 10^6$$

$$\Rightarrow k = \frac{1}{2}\ln 10^6 = 3\ln 10.$$

(b) Using separation of variables we find that

$$\begin{aligned}
& y' = 1 - y \\
\Rightarrow\ & \frac{dy}{dx} = -(y-1) \\
\Rightarrow\ & \int_3^y \frac{dy}{y-1} = \int_0^x -1\,dx \\
\Rightarrow\ & \ln(y-1) - \ln(3-1) = -x - (-0) \\
\Rightarrow\ & \ln(y-1) = \ln 2 - x \\
\Rightarrow\ & y(x) = 1 + e^{\ln 2 - x} = 1 + 2e^{-x}.
\end{aligned}$$

6. Let us first consider the corresponding homogeneous equation

$$y'' + 6y' + 5y = 0.$$

The corresponding characteristic equation is

$$r^2 + 6r + 5 = (r+5)(r+1) = 0$$

and so the roots are $r_1 = -5$ and $r_2 = -1$. Thus the general solution to the homogeneous equation is of the form

$$y(x) = C_1e^{r_1} + C_2e^{r_2} = C_1e^{-5x} + C_2e^{-x}.$$

For a particular solution we try

$$\begin{aligned}
& y_0(x) = A\cos(2x) + B\sin(2x) \\
\Rightarrow\ & y_0'(x) = 2(-A\sin(2x) + B\cos(2x)) \\
\Rightarrow\ & y_0''(x) = -4(A\cos(2x) + 4B\sin(2x))
\end{aligned}$$

Substituting in the original nonhomogeneous equation and collecting the common factors yields

$$\begin{aligned}
& y_0''(x) + 6y_0'(x) + 5y_0(x) = 145\sin(2x) \\
\Rightarrow\ & A\cos(2x) + B\sin(2x) + 12B\cos(2x) - 12A\sin(2x) = 145\sin(2x) \\
\Rightarrow\ & (A + 12B)\cos(2x) + (B - 12A)\sin(2x) = 145\sin(2x).
\end{aligned}$$

For all $x \in \mathbf{R}$ this equation is satisfied only if

$$\begin{cases} A + 12B &= 0 \\ B - 12A &= 145. \end{cases}$$

This solves to $A = -12$ and $B = 1$. So a particular solution to the nonhomogeneous equation is

$$y_0(x) = -12\cos(2x) + \sin(2x).$$

From the results above it follows that the general solution to the non-homogeneous equation is

$$\begin{aligned} y(x) &= C_1 e^{-5x} + C_2 e^{-x} + y_0(x) \\ &= C_1 e^{-5x} + C_2 e^{-x} - 12\cos(2x) + \sin(2x). \end{aligned}$$