

Aalto University School of Science  
Department of Mathematics and Systems Analysis

### Differential and Integral Calculus

Online exam **22.2.2021** at 13.00–17.00 (GMT+2)

**The best five (5) answers are counted.** You can solve all six problems, but only the best five are counted. Each problem is worth 6 points, so that the maximum is 30 points.

**Please remember to show the intermediate steps of your calculations!**

1. a) Find the sum of the series

$$\sum_{k=0}^{\infty} \frac{4^{k+1}}{5^{k-1}}.$$

- b) For which values of  $x$  is the sum

$$\sum_{k=1}^{\infty} \frac{k}{7^k} x^k$$

convergent?

2. a) Find an approximate value of  $\cos(\sin(0.1))$  by first replacing sine with its 3rd degree Maclaurin polynomial  $P_3(x)$  and then replacing cosine with its 2nd degree polynomial  $P_2(x)$ .

Note: Maclaurin polynomial = Taylor polynomial with respect to  $x_0 = 0$ .

- b) Show (e.g. L'Hospital's rule) that the limit

$$\lim_{x \rightarrow 0} \frac{e^{ax} - 1 + x^2}{\ln(1 + 2ax)}$$

does not depend on the value of the parameter  $a \neq 0$ .

3. Let  $f: \mathbf{R} \rightarrow \mathbf{R}$ ,  $f(x) = 3x + 4x^5$ .

a) Show that the function  $f$  is strictly increasing.

b) The function  $f$  is also surjective (onto), and therefore it has an inverse. Calculate  $(f^{-1})'(7)$ .

Hint: The inverse cannot be presented in terms of elementary functions, but the value  $f^{-1}(7)$  can be found by trial and error, or by solving a suitable equation with a calculator or mathematical software.

4. Calculate the improper integral

$$\int_0^{\infty} e^{-\sqrt{x}} dx$$

by first substituting  $x = t^2$ .

5. a) The value  $y(t)$  of a certain pyramid scheme at time  $t$  satisfies the differential equation

$$y' = ky, \quad k > 0 \text{ constant,}$$

and the initial condition  $y(0) = 100$  (euros). At time  $t = 2$  (years) the value has increased to  $y(2) = 10^8$  (euros). Determine the coefficient  $k$  using this information.

- b) Solve the differential equation  $y' = 1 - y$  with the initial condition  $y(0) = 3$ .
6. Find the general solution of the differential equation  $y'' + 6y' + 5y = 145 \sin(2x)$ .